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# Central charges, S-duality and massive vacua of $\mathcal{N} = 1^*$ super Yang-Mills

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## Abstract

We provide a simple derivation of the extremal values of the superpotential in massive vacua of  $\mathcal{N} = 1^*$  SYM, making use of the required modular weight for the central charge of BPS walls interpolating between these vacua. This modular weight descends from the action of S-duality on the  $\mathcal{N} = 4$  superalgebra which in turn is inherited from its classical action on the dyon spectrum. We show that this kinematic information, combined with minimal knowledge of the weak coupling asymptotics, is sufficient to determine the exact vacuum superpotentials in terms of Eisenstein series.

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Maximally supersymmetric  $\mathcal{N} = 4$  super Yang-Mills exhibits many remarkable features, among which its invariance under electric-magnetic duality is one of the most profound [1]. In the conformal phase this symmetry acts on the dimensionless coupling constant of the theory,

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d}, \quad \text{with } a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1, \quad (1)$$

and thus the coupling can be taken to lie in the fundamental domain of  $\text{SL}(2, \mathbb{Z})$ . If one moves away from the conformal point onto the Coulomb branch, this symmetry acts as a generator of the spectrum of massive BPS states. More precisely, on branches of the moduli space where only one of the three adjoint scalar fields  $\phi^a$  has a nonzero vev, the BPS spectrum [2],

$$M = |\mathcal{Z}|, \quad \mathcal{Z} = \sqrt{\frac{2}{\text{Im}\tau}} (n_e^a + \tau n_m^a) \phi_a, \quad (2)$$

is permuted by the action of  $\text{SL}(2, \mathbb{Z})$ , where

$$\begin{pmatrix} n_e \\ n_m \end{pmatrix} \longrightarrow \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \begin{pmatrix} n_e \\ n_m \end{pmatrix}. \quad (3)$$

This action of  $S$ -duality on the BPS spectrum is of course well-known. However, it has some interesting and less well-explored consequences. In particular, as shown by Intriligator [3] and recently discussed by Kapustin and Witten [4], there is an induced action on the supersymmetry algebra. For our purposes, it will be convenient to focus on the anticommutator of two left-handed supercharges in the  $\mathcal{N}$ -extended algebra, which contains two sets of allowed central charges,

$$\{Q_\alpha^A, Q_\beta^B\} = \epsilon_{\alpha\beta} Z^{[AB]} + \sigma_{\alpha\beta}^{\mu\nu} Z_{\mu\nu}^{(AB)}. \quad (4)$$

The first charge here appears only for  $\mathcal{N} \geq 2$ , and relates to the dyonic BPS spectrum considered above [2]. The second charge is supported by BPS domain walls [5, 6, 7] which, while not present in  $\mathcal{N} = 4$  SYM, do arise in many  $\mathcal{N} = 1$  theories to which  $\mathcal{N} = 4$  SYM flows under relevant perturbations.

In this note we will assume the exact invariance of  $\mathcal{N} = 4$  SYM under  $\text{SL}(2, \mathbb{Z})$  and explore the ensuing consequences for the tensorial central charges  $\mathcal{Z}_{\mu\nu}$  in the superalgebra. In particular, the required covariance of  $\mathcal{Z}_{\mu\nu}$  under modular transformations imposes stringent constraints on the low energy superpotentials in massive vacua which result from  $\mathcal{N} = 1$  perturbations of  $\mathcal{N} = 4$ . In particular, for the  $\mathcal{N} = 1^*$  deformation, we will be able to compute the exact vacuum superpotentials simply by requiring the necessary modular properties, given some cursory knowledge of the weak coupling asymptotics.

The primary constraint we require follows straightforwardly from the transformation rules above. In particular, we observe that the Lorentz-scalar central charge  $\mathcal{Z}$  transforms with modular weight,

$$w(\mathcal{Z}) = (-1/2, 1/2), \quad (5)$$

where the notation,  $w(f) = (w_1, w_2)$ , implies  $f((a\tau + b)/(c\tau + d)) = (c\tau + d)^{w_1} (c\bar{\tau} + d)^{w_2} f(\tau)$ . It is then apparent from the structure of the superalgebra that if we deform the theory with

a relevant perturbation, breaking  $\mathcal{N} = 4$  to  $\mathcal{N} = 1$  SUSY and leading to massive vacua and thus the possibility for BPS domain walls, the modular weight of the central charge  $\mathcal{Z}_{\mu\nu}$  must again be

$$w(\mathcal{Z}_{\mu\nu}) = (-1/2, 1/2). \quad (6)$$

This can be seen from the fact that the modular weight of the unbroken supercharges is already fixed from their embedding within the  $\mathcal{N} = 4$  algebra above.

Eq. (6) is the primary result that we will exploit in the remainder of this note. In particular, this central charge can generically be expressed in the form,

$$\mathcal{Z} = \Delta\mathcal{W}|_v, \quad (7)$$

with  $\mathcal{W}|_v$  the extremal value of the low energy superpotential in each vacuum between which the wall interpolates. Thus, we can conclude that the extrema of the superpotential (possibly corrected by a vacuum-independent constant) also inherit the same modular weight,

$$w(\mathcal{W}|_v) = (-1/2, 1/2). \quad (8)$$

The possibility for a vacuum-independent constant to be added so that  $\mathcal{W}$  and not just  $\Delta\mathcal{W}$  is modular can be associated with operator mixing [8].

If we now choose a particular relevant deformation of the theory, this result constitutes a powerful constraint on the extremal values of the superpotential. To proceed, we consider the  $\mathcal{N} = 1^*$  deformation, for which the classical superpotential takes the form,

$$\mathcal{W} = N\text{Tr} \left[ \Phi_1[\Phi_2, \Phi_3] + \frac{1}{2}m \sum_{i=1}^3 \Phi_i^2 \right]. \quad (9)$$

The exact extremal values for the superpotential in the massive vacua of this system were first obtained explicitly by Dorey [9] (see also [10, 11, 12, 8, 13, 14]) and are well-known, but we will illustrate this technique by rederiving these results in a very straightforward manner which will also serve to illustrate the extent to which these results are determined purely by kinematics. Indeed, we will limit the dynamical input to knowledge of the classical Higgs vacuum, which is given by expressing  $\Phi_i$  in terms of the unique irreducible representation of  $\text{SU}(2)$  of dimension  $N$  (taking the gauge group to be  $\text{SU}(N)$ ). In the normalization above, we have

$$\mathcal{W}|_h^{\text{cl}} = \frac{N^3}{24} m^3 [(-N) + \mathcal{O}(1) + \mathcal{O}(e^{2\pi i N \tau})], \quad (10)$$

which is valid in the weak coupling regime,  $\tau \rightarrow i\infty$ . The  $\mathcal{O}(N^3)$  prefactor results directly from our normalization of the superpotential. We have retained only the leading constant term for large  $N$ , as this allows us to exclude possible operator mixing ambiguities which vanish at large  $N$ . We have also exhibited the scaling of the leading nonperturbative correction, an  $N$ -instanton contribution. The origin of this scaling follows first of all from the fact that, since this vacuum can be reliably placed at weak coupling, we expect the nonperturbative contributions to be exhausted by instantons. Secondly, the absence of  $k$ -instanton contributions for  $k < N$  is most clearly understood from an analysis of the ADHM constraints, or

more directly from the realization of the relevant instanton configurations in terms of a D(-1)-D3 system. For instantons to contribute to the superpotential, the worldvolume theory of the D(-1)-branes must have a supersymmetric vacuum and, as shown in [15], in the presence of the  $\mathcal{N} = 1^*$  deformation the  $F$ -term constraints are only satisfied if the number of instantons is a multiple of  $N$ . We are thus led to the above scaling of semi-classical contributions in the Higgs vacuum. These results will be of use below.

To proceed in making use of the general constraint (8), we need to determine the modular weight of the adjoint mass parameter  $m$ . This can be done by returning to the conformal phase and requiring that the chiral primaries be modular invariant, from which it follows that [8]

$$w(m) = (-5/6, 1/6). \quad (11)$$

Since the effective superpotential in each vacuum must vanish as the deformation  $m \rightarrow 0$ , we can use global symmetries and dimensional analysis to write

$$\mathcal{W}|_v = \frac{N^3}{24} m^3 X(\tau)|_v, \quad (12)$$

with an unknown function  $X(\tau)$ , depending only on the (bare) coupling  $\tau$ . By holomorphy of the superpotential it follows that  $X$  must be a holomorphic modular form, and indeed using (11) we see that

$$w(X) = (2, 0), \quad (13)$$

for consistency with the modular weight of the central charge.

This leads us to conclude that the nontrivial  $\tau$ -dependence of the vacuum condensates in any massive vacuum of  $\mathcal{N} = 1^*$  SYM must be determined by a suitable holomorphic weight-2 modular form of  $\text{SL}(2, \mathbb{Z})$ . Unfortunately, this neat conclusion cannot be correct as there are no such forms. There is a unique candidate which comes closest, namely the regulated second Eisenstein series,

$$E_2(\tau) \equiv \frac{3}{\pi^2} \sum_{(a,b) \in \mathbb{Z}^2 - \{0,0\}} \frac{1}{(a\tau + b)^2}, \quad (14)$$

which however transforms under  $\tau \rightarrow -1/\tau$  as weight-2 only up to an additive shift.

In fact, this conclusion shouldn't be a surprise as the dyonic central charge  $\mathcal{Z}(\tau)$  is in fact only a modular form of weight  $(-1/2, 1/2)$  up to *permutation*, as a suitable action on the electric and magnetic charges,  $(n_e, n_m)$ , was also required. Therefore, we should anticipate a similar structure in the present case. Indeed, physically we expect these massive vacua to be associated with the condensation of various dyonic states, and thus should directly inherit this permutation under  $\text{SL}(2, \mathbb{Z})$ .

It follows that a given massive vacuum will only preserve a specific subgroup  $\Gamma \subset \text{SL}(2, \mathbb{Z})$ , and the broken generators will induce the required permutation. We could proceed by trying to determine the precise subgroup  $\Gamma$ ; the required forms will be of weight-2 with respect to this subgroup. However, it is clear that a priori there is no reason to believe that all massive vacua preserve the same subgroup, or equivalently that all vacua lie on the same orbit of the associated coset. Instead we will construct the full orbit explicitly without making

an assumption about the residual subgroup in any given vacuum. This clearly requires some dynamical information, but it will be sufficient to use knowledge of the weak coupling asymptotics of the Higgs vacuum discussed above.

To see how this works, recall that weight-2 modular forms for general (congruence) subgroups of  $\text{SL}(2, \mathbb{Z})$  can be constructed in terms of the basis of forms for the principal congruence subgroups (see e.g. [16]),

$$\Gamma(M) \equiv \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}), \quad a = d = 1 \pmod{M}, \quad b = c = 0 \pmod{M} \right\}. \quad (15)$$

which (for weight-2) are suitable linear combinations of ‘level  $M$ ’ Eisenstein series

$$G_2^{\vec{v}}(\tau) \equiv \sum_{(a,b)=(\vec{v} \bmod M) \in \mathbb{Z}^2 - \{0,0\}} \frac{1}{(a\tau + b)^2}, \quad (16)$$

which make use of a vector  $\vec{v} = (v_1, v_2)$  with  $1 \leq v_i \leq M$ .

The vectors  $\vec{v}$  are in one-to-one correspondence with the cusps of  $\Gamma \subset \text{SL}(2, \mathbb{Z})$ . Note that although  $G_2(\tau)$  is not a strict modular form itself, the additive shift under  $\tau \rightarrow -1/\tau$  can be cancelled by taking a suitable linear combination, given a subgroup  $\Gamma$  with at least two cusps.

For a fixed vector  $\vec{v}$ , generic elements of  $\text{SL}(2, \mathbb{Z})$  will permute the basis forms via the natural action

$$\vec{v} \rightarrow \{\vec{v}\gamma | \gamma \in \text{SL}(2, \mathbb{Z})\}. \quad (17)$$

To pin down this action precisely, we need to find one point on each orbit. As noted above, it will be sufficient to use the Higgs vacuum which is visible at weak coupling. Expanding the Eisenstein series in the weak coupling  $\tau \rightarrow i\infty$  limit, we find

$$G_2^{\vec{v}} \longrightarrow c_1 \delta_{v_1 0} + \frac{c_2}{M^2} e^{2\pi i(\tau n + v_2 m)/M} + \dots, \quad (18)$$

where  $c_1$  and  $c_2$  are constants that will not be important in what follows. The leading nonperturbative correction shown here is determined by the constraint on the integers  $n \geq 1$  and  $m|n$ , namely that  $n/m \equiv v_1 \pmod{M}$ . Matching the asymptotics of the Higgs vacuum,  $X(\tau)|_h \rightarrow (-N) + \mathcal{O}(1) + \mathcal{O}(e^{2\pi i N \tau})$ , requires  $\vec{v} = (0, v_2)$  in order to retain the constant term. In this case,  $n/m \equiv 0 \pmod{M}$ , and thus to ensure that the leading nonperturbative correction is no larger than the  $N$ -instanton factor, we need to set  $M = N$ , i.e. given by the rank of the gauge group, and sum over all the allowed values of  $v_2$ . Up to this point  $M$  was simply a parameter labelling the allowed set of modular forms, but with hindsight a relation to the rank of the gauge group is perfectly natural in the sense that fully Higgsed vacua, which are visible at weak coupling, are unique for each  $\text{SU}(N)$  gauge group.

We are thus led to a unique possibility:

$$\frac{3}{\pi^2} \sum_{v_2=0}^{N-1} G_2^{(0, v_2)} = E_2(N\tau). \quad (19)$$

This combination is not modular, but a simple linear combination which cancels the additive shift under  $\tau \rightarrow -1/\tau$  is given by

$$E_2(N\tau) - \frac{1}{N}E_2(\tau). \quad (20)$$

To match the constant asymptotic value at large  $N$ , we can fix the prefactor and identify,

$$X(\tau)|_h = E_2(\tau) - NE_2(N\tau), \quad (21)$$

which indeed coincides with the exact expression for the superpotential in the Higgs vacuum, determined previously using other approaches [9, 11, 13].

Alternatively, this result could have been deduced by ‘guessing’ that the Higgs vacuum, since it is unique, would preserve the largest congruence subgroup. Identifying the level with the rank  $N$  as above, these subgroups are given by,

$$\Gamma_0(N) \equiv \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}), \quad c = 0 \pmod{N} \right\}. \quad (22)$$

For a given prime integer  $N$ , these subgroups have precisely two cusps and therefore a *unique* modular form of weight 2, which is conventionally expressed in terms of the second Eisenstein series in precisely the combination deduced above [16],

$$E_{2,N}(\tau) = NE_2(N\tau) - E_2(\tau). \quad (23)$$

We can now determine the remaining massive vacua from the orbit of the Higgs vacuum under  $\text{SL}(2, \mathbb{Z})$ . The sum in (19) ensures that the subgroup  $\Gamma$  preserved by the vacuum is of index  $N$  within  $\text{SL}(2, \mathbb{Z})$ , and we will henceforth assume that all massive vacua lie on the orbit of the Higgs vacuum. The action of  $\text{SL}(2, \mathbb{Z})$  will preserve this index, and thus it is straightforward to write down the remaining possibilities, which are given by summing over  $v_1$  and  $v_2$  with fixed index  $N$ . i.e., for  $N = pq$ ,

$$\frac{3}{\pi^2} \sum_{v_1=0}^{q-1} \sum_{v_2=0}^{p-1} G_2^{(pv_1, qv_2)} = \frac{1}{q^2} E_2 \left( \frac{p\tau}{q} \right). \quad (24)$$

Taking the same linear combination to restore the modular transformation properties, we obtain the candidate vacua,

$$X(\tau)|_{p,q} = E_2(\tau) - \frac{p}{q} E_2 \left( \frac{p\tau}{q} \right), \quad (25)$$

which (with the shifts  $\tau \rightarrow \tau + k$ ,  $k = 0..q-1$ , induced by  $2\pi$ -rotations of the UV  $\theta$  parameter) reproduce all the known massive vacua of  $\mathcal{N} = 1^*$  SYM [10, 9, 11, 12, 8]. For  $N$  prime, the choice  $(p, q) = (N, 1)$  reproduces the Higgs vacuum discussed above, while the alternative  $(p, q) = (1, N)$  has the appropriate scaling to be identified with gaugino condensation in confining vacua. In general, the appearance of the vector  $\vec{v}$  is natural here in the context of ‘t Hooft’s  $\mathbb{Z}_N \times \mathbb{Z}_N$  classification of massive phases [17, 18], as it inherits a natural action of

this group from the remaining group elements of  $\text{SL}(2, \mathbb{Z})$  which thus permute all the massive phases.

One should bear in mind that the procedure we have followed can in principle only determine the relative differences between superpotentials in different vacua. However, in the present case, this ambiguity has been fixed by the additional assumption of modular covariance of  $\mathcal{W}$ , rather than just  $\Delta\mathcal{W}$ .

We will finish with some additional remarks on these results.

- The orbit of the Higgs vacuum under the broken generators of  $\text{SL}(2, \mathbb{Z})$  has an interesting interpretation in terms of the Hecke  $T_N$  operators, which map the space of weight- $k$  forms into itself by a suitable averaging procedure. More precisely, if  $f$  is a modular form of  $\text{SL}(2, \mathbb{Z})$ , the action of the Hecke operator  $T_N$  is given by (see e.g. [16])

$$T_N f(\tau) = \sum_{pq=N, k=1..N} \frac{p}{q} f\left(\frac{p\tau + k}{q}\right), \quad (26)$$

with the sum over the divisors  $p$  of  $N$  and  $k = 1..N$ . The forms that are relevant here are eigenvectors of  $T_N$  of weight-2, of which there is only one, the regulated second Eisenstein series, and

$$T_N E_2(\tau) = \sigma_1(N) E_2(\tau), \quad (27)$$

where  $\sigma_1(N) = \sum_{d|N} d$  sums the divisors of  $N$ . Thus we can identify the orbit of the Higgs vacuum under  $\text{SL}(2, \mathbb{Z})$  with the orbit generated by  $T_N$ . Indeed, this correspondence is less mysterious given that both averages must restore full  $\text{SL}(2, \mathbb{Z})$  covariance.

- In recent work, Kapustin and Witten [4] pointed out a relation between specific geometric Hecke operators and the insertion of 't Hooft operators. Since the latter also provide a shift of the vacuum analogous to the action of  $\text{SL}(2, \mathbb{Z})$ , it would be interesting to understand the relation to this aspect of [4] in more detail.
- There is an alternative definition of the action of the Hecke operator  $T_N$  using lattices. In particular, if the lattice associated with  $\text{SL}(2, \mathbb{Z})$  has periods  $(\omega_1, \omega_2)$  with  $\tau = \omega_2/\omega_1$ , then the Hecke operator acts by summing over all sublattices of index  $N$ . This is precisely the picture of massive vacua in  $\mathcal{N} = 1^*$  that emerges from perturbations of  $\mathcal{N} = 2$  SYM [10] and also compactification on  $\mathbb{R}^3 \times S^1$  [9].

In conclusion, the approach we have outlined is quite general, and it would be interesting to know whether it can usefully be applied to deduce the vacuum superpotentials for other relevant perturbations of  $\mathcal{N} = 4$  SYM, or indeed perturbations of other conformal theories, such as  $\mathcal{N} = 2$  SQCD with  $N_f = 2N_c$ ,  $\beta$ -deformations of  $\mathcal{N} = 4$ , and other examples which are believed to admit an action of  $\text{SL}(2, \mathbb{Z})$ .

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